Inherent Automorphism and Q-Conjugacy Character Tables of Finite Groups. An Application to Combinatorial Enumeration of Isomers

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Q-Conjugacy character tables are proposed for finite groups and applied to combinatorial enumeration. Thus, the maturity of an irreducible representation is related to the maturity of a finite group by means of the relationship between the inherent automorphism of the group and its inner portion. As a result, a character table is transformed into a more concise form called a **Q**-conjugacy character table. Matured characters are defined as dominant-class functions on the basis of such a **Q**-conjugacy character table. Thereby, a matured character is represented by a linear combination of **Q**-conjugacy characters. By starting from **Q**-conjugacy character tables, characteristic monomial tables for finite groups are obtained. They are applied to combinatorial enumeration of isomers.

One type of group-theoretical approaches to chemistry has been concerned with such diverse chemical fields as theoretical, organic, inorganic, and other chemistries,[1—8] where linear representations and irreducible representations are keys for analyzing various chemical phenomena. Thus, characters and irreducible characters (collected in character tables) are used as versatile tools. Another type of group-theoretical approaches to chemistry has aimed at chemical combinatorics[9—20] where permutation representations and coset representations play an important role in solving combinatorial problems. The counterparts of the characters and irreducible characters are marks (fixed-point vectors) and marks of coset representations (collected in mark tables).

In previous reports,[21,22] we have shown that, although the two types of approaches have their own territories, there are some common fields which permit both types of approaches. Such overlapped areas concerning linear and permutation representations have been characterized in terms of the following concepts:

Linear representation	Permutation representation
matured representation	(non-)dominant representation
Q-conjugacy representation	dominant representation
markaracter	markaracter
Q-conjugacy character	dominant markaracter
Q-conjugacy character table	(dominant) markaracter table
characteristic monomial	dominant USCI
characteristic monomial table	dominant USCI table

As shown in this table, a common basis for the present approach is the concept of markaracter (mark-character). A markaracter has been originally defined for a permutation representation as a dominant-class function based on dominant classes.[21,22] Note that the usual character for a linear representation is a class function, which stems from conjugacy classes. Such a markaracter has been shown to be represented by a linear combination of dominant markaracters, while a usual character is represented by a linear combination of irreducible characters.

For cyclic groups, the concept of markaracter has been extended to treat linear representations by redefining dominant classes as **Q**-conjugacy classes.[23] Thus, after the definition of **Q**-conjugacy characters as dominant-class functions for cyclic groups, a markaracter has been shown to be represented alternatively by a linear combination of such **Q**-conjugacy characters.

By formulating the maturity of finite groups, the concept of markaracter has been extended to treat matured groups of finite order.[23] Although the treatment has been applied to unmatured groups, it has been insufficient because we have taken account only of (non-)dominant representations as matured representations. In particular, **Q**-conjugacy characters have not been defined for characterizing finite groups in general.

In order to have a comprehensive perspective, we shall treat unmatured groups as general cases, where matured representations other than (non-)dominant representations should be investigated as a first target. This target inevitably requires well-defined concept of **Q**-conjugacy characters for finite groups, which will be discussed in terms of inherent automorphism. The second target is the extension of characteristic monomials so as to be applicable to combinatorial enumeration based on matured and unmatured groups of finite order.

1 Theoretical Foundations. Inherent Automorphism and Related Concepts

1.1 Problem Setting. We have already indicated that character tables can be converted into **Q**-conjugacy character tables for cyclic groups.[23] However, we are unware of whether such conversion is available or not for finite groups. Hence, one of the purposes of the present paper is to clarify the availability of conversion for any finite groups.

Let us consider the character table of $\mathbf{D}_5$ (Table 1, left) cited from Ref. 1. Since each character is a class function, the character table (as a matrix) is a 4×4 square matrix in accord with the number of conjugacy classes. The table contains the values, $2\cos 72°$ and $2\cos 144°$, which come from the 5th primitive root of unity ($\varepsilon = \exp(2\pi/5)$). According to the unmaturity of the group $\mathbf{D}_5$,[23] each of the irreducible characters (E_1 and E_2) contains the different values in the $\mathbf{K}_{31}$ ($2C_5$) and the $\mathbf{K}_{32}$ ($2C_5^2$) columns.

By inspection of Table 1 (left), we are able to show that the summation of the irreducible characters (E_1 and E_2) gives a character that has the same values in the $\mathbf{K}_{31}$ and the $\mathbf{K}_{32}$ columns because of $2\cos 72°+2\cos 144° = \varepsilon+\varepsilon^2+\varepsilon^3+\varepsilon^4 = -1$. Thereby, a character ($E_1+E_2$) as a class function is converted into a **Q**-conjugacy character (E) as a **Q**-conjugacy-class (dominant-class) function. Hence, the reduction of columns as well as that of rows gives Table 1 (right) as a 3×3 square matrix, which is called a **Q**-conjugacy character table. A problem to be solved is whether or not the conversion of Table 1 (left) into Table 1 (right) has sound reasoning. For such reasoning, we shall clarify the relationship between

1. The maturity of a **Q**-conjugacy class, which is concerned with the column-reduction of a character table into a **Q**-conjugacy character table, and
2. the maturity of a representation (or character), which is concerned with the row-reduction of a character table into a **Q**-conjugacy character table.

Since the first item has already been discussed,[23] the second item is the target of the present paper, where the concept of inherent automorphism is the foundation of the reasoning used here.

1.2 Automorphism of Cyclic Groups. Let us consider a cyclic group **H**. It is well known that Aut **H** (the automorphism group of **H**) is an abelian group of order $\varphi(|\mathbf{H}|)$ and Inn **H** (the inner automorphism group of **H**) is an identity group (Theorem A11.3 of Ref. 24). Let $\tilde{h}$ be a generator of the cyclic group **H**, i.e., $\langle\tilde{h}\rangle = \mathbf{H}$. Then, we obtain the group **H** as an ordered set represented by

$$\mathbf{H} = \{\tilde{h}^1, \tilde{h}^2, \cdots, \tilde{h}^r, \cdots, \tilde{h}^n (= I)\}, \tag{1}$$

where $n = |\mathbf{H}|$. When an integer s is selected to be coprime to n, the element $\tilde{h}^s$ also generates the group **H**, which is regarded as another ordered set, i.e.,

$$\mathbf{H}^{(s)} = \{(\tilde{h}^s)^1, (\tilde{h}^s)^2, \cdots, (\tilde{h}^s)^r, \cdots, (\tilde{h}^s)^n (= I)\}, \tag{2}$$

It should be noted that the sets **H** and $\mathbf{H}^{(s)}$ are identical with each other when we take no account of the alignments of elements. Moreover, they are closely related even when the alignments of elements are taken into consideration. Since the elements $\tilde{h}^r$ ($r = 1, 2, \cdots, n$) of **H** correspond to the elements $(\tilde{h}^s)^r = (\tilde{h}^r)^s$ of $\mathbf{H}^{(s)}$ in one-to-one fashion, the latter set $\mathbf{H}^{(s)}$ can be created by permuting the elements of the original set **H** (Eq. 1). Hence, the correspondence $\tilde{h}\to\tilde{h}^s$ provides the permutation $P^{(s)}$ represented by

$$P^{(s)} = \begin{pmatrix} \mathbf{H} \\ \mathbf{H}^{(s)} \end{pmatrix} = \begin{pmatrix} \tilde{h}^1 & \tilde{h}^2 & \cdots & \tilde{h}^n \\ (\tilde{h}^s)^1 & (\tilde{h}^s)^2 & \cdots & (\tilde{h}^s)^n \end{pmatrix}. \tag{3}$$

The transformation of **H** to $\mathbf{H}^{(s)}$ represented by the permutation $P^{(s)}$ keeps the corresponding multiplication table invariant. Obviously, $P^{(1)}$ is the identity permutation. Let **P** be the set of permutations $P^{(s)}$ defined by Eq. 3, where s runs so as to satisfy $(s,n) = 1$:

$$\mathbf{P} = \{P^{(s)} | (s,n) = 1\}. \tag{4}$$

It follows that **P** is shown to be a group (Appendix A) whose order is equal to $|\mathbf{P}| = \varphi(n)$.

1.3 Inherent Automorphism of Finite Groups. Suppose that **G** is a finite group and that **H** represented by Eq. 1 is a cyclic subgroup of **G**. Let $\mathbf{H}x$ and $\mathbf{H}x'$ be cosets contained in a coset decomposition of **G** by **H**:

$$\mathbf{G} = \mathbf{H} + \cdots + \mathbf{H}x + \cdots + \mathbf{H}x' + \cdots. \tag{5}$$

Consider a product $(\tilde{h}^a x)(\tilde{h}^b x')$ produced from two elements, $\tilde{h}^a x \in \mathbf{H}x$ and $\tilde{h}^b x' \in \mathbf{H}x'$. Since x and $\tilde{h}$ appearing in the product are not always commutative, we place $x\tilde{h} = hx$ or equivalently

$$\tilde{h} = x^{-1}hx, \tag{6}$$

where $h \in x\mathbf{H}x^{-1}$. We easily obtain

$$\tilde{h}^a = x^{-1}h^a x, \text{ and } \tilde{h}^b = x^{-1}h^b x, \tag{7}$$

where a and b are integers. Thereby, the product is transformed as follows:

$$(\tilde{h}^a x)(\tilde{h}^b x') = (\tilde{h}^a x)(x^{-1}h^b x)x' = \tilde{h}^a h^b x x'. \tag{8}$$

Let us consider a transformation of the generator $\tilde{h}$ to another one $\tilde{h}^s$, where s is coprime to n (Eq. 2). Then, we obtain the corresponding cosets, $\mathbf{H}^{(s)}x$ and $\mathbf{H}^{(s)}x'$. It follows that two elements, $\tilde{h}^a x \in \mathbf{H}^{(s)}x$ and $\tilde{h}^b x' \in \mathbf{H}^{(s)}x'$, give a product,

$$((\tilde{h}^s)^a x)((\tilde{h}^s)^b x') = (\tilde{h}^{sa} x)(x^{-1}h^{sb}x)x' = \tilde{h}^{sa}h^{sb}xx' = (\tilde{h}^s)^a (h^s)^b xx'. \tag{9}$$

Table 1. Character Table (left) and **Q**-Conjugacy Character Table (right) for $\mathbf{D}_5$

	$\mathbf{K}_1$	$\mathbf{K}_2$	$\mathbf{K}_{31}$	$\mathbf{K}_{32}$
	I	$5C_2$	$2C_5$	$2C_5^2$
A_1	1	1	1	1
A_2	1	-1	1	1
E_1	2	0	$2\cos 72°$	$2\cos 144°$
E_2	2	0	$2\cos 144°$	$2\cos 72°$

	$\mathbf{C}_1$	$\mathbf{C}_2$	$\mathbf{C}_5$
A_1	1	1	1
A_2	1	-1	1
E	4	0	-1

Equations 8 and 9 show that the transformation of **G** into itself (in terms of the transformation of $\mathbf{H}x$ into $\mathbf{H}^{(s)}x$) is a homomorphism.

The discussion described in the preceding paragraph can be made more sophisticated as follows. Let **N** be the normalizer of **H** within **G**, i.e., $\mathbf{N} = \mathbf{N}_{\mathbf{G}}(\mathbf{H})$. Then we have coset decompositions:

$$\mathbf{G} = \mathbf{N}_{g_1} + \mathbf{N}_{g_2} + \cdots \mathbf{N}_{g_a}, \tag{10}$$

$$\mathbf{N} = \mathbf{H}t_1 + \mathbf{H}t_2 + \cdots + \mathbf{H}t_b. \tag{11}$$

The representatives of every cosets are collected to give transversals:

$$\mathbf{A} = \{g_1, g_2, \cdots, g_a\} \tag{12}$$

$$\mathbf{B} = \{t_1, t_2, \cdots, t_b\} \tag{13}$$

Suppose that the cyclic group **H** represented by Eq. 1 is a cyclic subgroup of the group **G**, which is characterized by Eqs. 10 and 11. Select an element g from the transversal **A** (Eq. 12) and an element t from the traversal **B** (Eq. 13). Then we have a coset $\mathbf{H}tg$ ($\subset \mathbf{N}_g$).

$$\mathbf{H}tg = tg[g^{-1}(t^{-1}\mathbf{H}t)g] = tg[g^{-1}\mathbf{H}^t g] \tag{14}$$

where the inner set $\mathbf{H}^t$ is represented by

$$\mathbf{H}^t = t^{-1}\mathbf{H}t = \{t^{-1}\tilde{h}t, t^{-1}\tilde{h}^2 t, \cdots, t^{-1}\tilde{h}^n t\}, \tag{15}$$

which is equivalent to **H** without considering the alignment of elements. Since $t^{-1}\tilde{h}t$ is also a generator of **H**, we are able to put $t^{-1}\tilde{h}t = \tilde{h}^r$, where the integer r is coprime to n. Then, the ordered set $\mathbf{H}^t$ (Eq. 15) is rewritten to be

$$\mathbf{H}^t = t^{-1}\mathbf{H}t = \{(\tilde{h}^r)^1, (\tilde{h}^r)^2, \cdots, (\tilde{h}^r)^n\}. \tag{16}$$

By virtue of Eq. 14, the ordered set **H** with the generator $\tilde{h}$ (Eq. 1) now corresponds to the ordered set $g^{-1}\mathbf{H}^t g$ with the generator $g^{-1}\tilde{h}^r g$.

Let us select another element $\tilde{g}$ from the transversal **A** (Eq. 12) and an element $\tilde{t}$ from the transversal **B** (Eq. 13). Then we have a coset $\mathbf{H}\tilde{t}\tilde{g}$ ($\subset \mathbf{N}\tilde{g}$). For an element $\tilde{h}^a tg \in \mathbf{H}tg$ and another element $\tilde{h}^b\tilde{t}\tilde{g} \in \mathbf{H}\tilde{t}\tilde{g}$, we take account of their product, i.e.,

$$(\tilde{h}^a tg)(\tilde{h}^b\tilde{t}\tilde{g}) = tg[g^{-1}t^{-1}\tilde{h}^a tg]\tilde{h}^b\tilde{t}\tilde{g} = tg[g^{-1}t^{-1}\tilde{h}tg]^a\tilde{h}^b\tilde{t}\tilde{g}. \tag{17}$$

We next consider a transformation of $\tilde{h}$ into $\tilde{h}^s$ to produce the permutation represented by Eq. 3. As a result, the elements $\tilde{h}^a tg$ and $\tilde{h}^b\tilde{t}\tilde{g}$ are transformed into $\tilde{h}^{sa}tg$ and $\tilde{h}^{sb}\tilde{t}\tilde{g}$, the product of which is represented by

$$(\tilde{h}^{sa}tg)(\tilde{h}^{sb}\tilde{t}\tilde{g}) = tg[g^{-1}t^{-1}\tilde{h}tg]^{sa}\tilde{h}^{sb}\tilde{t}\tilde{g}. \tag{18}$$

The product given by Eq. 17 corresponds to that of Eq. 18, since we have

$$\begin{pmatrix} g^{-1}t^{-1}\mathbf{H}tg \\ g^{-1}t^{-1}\mathbf{H}^{(s)}tg \end{pmatrix} = \begin{pmatrix} g^{-1}t^{-1}\tilde{h}^1 tg & g^{-1}t^{-1}\tilde{h}^2 tg & \cdots & g^{-1}t^{-1}\tilde{h}^n tg \\ g^{-1}t^{-1}(\tilde{h}^s)^1 tg & g^{-1}t^{-1}(\tilde{h}^s)^2 tg & \cdots & g^{-1}t^{-1}(\tilde{h}^s)^n tg \end{pmatrix} = P^{(s)}, \tag{19}$$

where the $P^{(s)}$ has appeared in Eq. 3.

Let us regard the coset $\mathbf{H}tg$ as an ordered set. When we take account of the permutation $P^{(s)}$ (Eq. 19), we have the corresponding permutation concerning the coset:

$$L_{tg}^{(s)} = \begin{pmatrix} \mathbf{H}tg \\ \mathbf{H}^{(s)}tg \end{pmatrix} = \begin{pmatrix} \tilde{h}^1 tg & \tilde{h}^2 tg & \cdots & \tilde{h}^n tg \\ (\tilde{h}^s)^1 tg & (\tilde{h}^s)^2 tg & \cdots & (\tilde{h}^s)^n tg \end{pmatrix}, \tag{20}$$

which is the same permutation as $P^{(s)}$ in an abstract fashion in the light of Eq. 19.

When the elements t and g respectively run over the transversals of the coset decomposition (Eqs. 12 and 13), it produces an automorphism represented by $L^{(s)}$. This discussion is summarized as a definition:

Definition 1. (Inherent Automorphism) *Let the symbol $L^{(s)}$ represent the permutation constructed by using all of the permutations $L_{tg}^{(s)}$ (Eq. 20) where t and g run over the transversals concerning Eqs. 12 and 13. We call the resulting permutation $L^{(s)}$ an inherent automorphism.*

Note that the inherent automorphism depends upon the selection of the transversals **A** (Eq. 12) and **B** (Eq. 13). Since the integer s runs over the integers coprime to n, the set of all the permutations $L^{(s)}$ forms a group **L**,

$$\mathrm{Inh}_{\mathbf{H}}(\mathbf{G}) = \mathbf{L} = \{L^{(s)} | (s,n) = 1\} \tag{21}$$

which is isomorphic to **P** described in Eq. 4. As a result, we obtain a theorem:

Theorem 1. (The Inherent Automorphism Group) *Let* **H** *be a cyclic subgroup of* **G**. *All of the permutations $L^{(s)}$ defined in Def. 1 form a group* **L** *of order $\varphi(n)$, where the integer s runs so as to satisfy $(s,n) = 1$. We call the group* **L** *'the inherent automorphism group of* **G** *concerning the cyclic subgroup* **H**' *or simply 'the* **H***-inherent automorphism group of* **G**', *which is designated by the symbol* $\mathrm{Inh}_{\mathbf{H}}(\mathbf{G})$. *The order of* $\mathrm{Inh}_{\mathbf{H}}(\mathbf{G})$ *is* $\varphi(|\mathbf{H}|)$.

Obviously, the **H**-inherent automorphism group of **G** ($\mathrm{Inh}_{\mathbf{H}}(\mathbf{G})$) is a subgroup of Aut **G**. Theorem 1 shows that the automorphism group of a cyclic subgroup **H** (Aut **H**) can be related to the group $\mathrm{Inh}_{\mathbf{H}}(\mathbf{G})$ by means of Eq. 20, where the two automorphism groups are isomorphic, i.e. $\mathrm{Inh}_{\mathbf{G}} \cong$ Aut **H**.

1.4 Inner Automorphisms of a Cyclic Subgroup. Although most of the discussions in this subsection are well-known, minimal results concerning a cyclic subgroup are revisited for further discussions.

Let $\mathbf{C}_{\mathbf{N}}(\tilde{h})$ denote the centralizer of $\tilde{h} \in \mathbf{N}$, where $\tilde{h}$ is a generator of a cyclic subgroup **H** and **N** ($= \mathbf{N}_{\mathbf{G}}(\mathbf{H})$) is the normalizer of **H** within **G**. Then, we can regard $\mathbf{C}_{\mathbf{N}}(\tilde{h})$ as the centralizer of **H** within **N**; this is denoted by the symbol $\mathbf{C}_{\mathbf{N}}(\mathbf{H})$. Let us consider a double coset decomposition:

$$\begin{aligned}\mathbf{N} &= \mathbf{C}_{\mathbf{N}}(\mathbf{H})t_1\mathbf{H} + \mathbf{C}_{\mathbf{N}}(\mathbf{H})t_2\mathbf{H} + \cdots + \mathbf{C}_{\mathbf{N}}(\mathbf{H})t_q\mathbf{H} \\ &= \mathbf{C}_{\mathbf{N}}(\mathbf{H})t_1 + \mathbf{C}_{\mathbf{N}}(\mathbf{H})t_2 + \cdots + \mathbf{C}_{\mathbf{N}}(\mathbf{H})t_q\end{aligned} \tag{22}$$

where the transversal is selected appropriately from **B**:

$$\mathbf{B}' = \{t_1, t_2, \cdots, t_q\}, \tag{23}$$

and q is calculated to be

$$q = \frac{|\mathbf{N_G(H)}|}{|\mathbf{C_N(H)}|}. \tag{24}$$

We here select t_i from $\mathbf{B}'$ (Eq. 23). Let $\tilde{h}$ be a generator of $\mathbf{H}$, the order of which is equal to n. The definition of the normalizer gives $t_i^{-1}\mathbf{H}t_i = \mathbf{H}$. Hence, the transversal given by Eq. 23 provides the following relationships:

$$t_i^{-1}\tilde{h}t_i = \tilde{h}^{r_i} \tag{25}$$

for $i = 1, 2\cdots, q$, where the symbol r_i denotes an integer coprime to n (i.e., $(r_i, n) = 1$). Note that ct_i for $c \in \mathbf{C}$ satisfies

$$t_i^{-1}c^{-1}\tilde{h}ct_i = t_i^{-1}(c^{-1}\tilde{h}c)t_i = t_i^{-1}\tilde{h}t_i = \tilde{h}^{r_i}. \tag{26}$$

We revisit the normalizer $\mathbf{N_G(H)}$ for a cyclic subgroup $\mathbf{H}$ of $\mathbf{G}$. Let t_i be selected from $\mathbf{B}'$, which produces the corresponding ordered set $\mathbf{H}^{t_i}$ (see Eq. 25). Thereby, the correspondence $\tilde{h} \to t_i^{-1}\tilde{h}t_i$ gives a permutation:

$$P_{t_i} = \begin{pmatrix} \mathbf{H} \\ \mathbf{H}^{t_i} \end{pmatrix} = \begin{pmatrix} \tilde{h}^1 & \tilde{h}^2 & \cdots & \tilde{h}^n \\ t_i^{-1}\tilde{h}t_i & t_i^{-1}\tilde{h}^2t_i & \cdots & t_i^{-1}\tilde{h}^nt_i \end{pmatrix} = \begin{pmatrix} \tilde{h}^1 & \tilde{h}^2 & \cdots & \tilde{h}^n \\ (\tilde{h}^{r_i})^1 & (\tilde{h}^{r_i})^2 & \cdots & (\tilde{h}^{r_i})^n \end{pmatrix} \tag{27}$$

for $i = 1, 2, \cdots, q$. The transformation of $\mathbf{H}$ to $t_i^{-1}\mathbf{H}t_i$ represented by the permutation P_{t_i} is an automorphism of $\mathbf{H}$.

When t_i runs over the transversal $\mathbf{B}'$ (Eq. 23), the automorphism P_{t_i} gives a set $\mathbf{P}'$.

$$\mathbf{P}' = \{P_{t_i} | t_i \in \mathbf{B}'\}. \tag{28}$$

Since we have $t_\beta^{-1}(t_\alpha^{-1}\tilde{h}t_\alpha)t_\beta = (t_\alpha t_\beta)^{-1}\tilde{h}(t_\alpha t_\beta)$ for all $t_\alpha, t_\beta \in \mathbf{B}'$, we have

$$P_{t_\beta}P_{t_\alpha} = P_{t_\alpha t_\beta}.$$

As a result, the set $\mathbf{P}'$ is a group. Since the order q is obtained in virtue of Eq. 24, we have a theorem:

Theorem 2. *Let* $\mathbf{P}'$ *be defined by Eq. 28. Then, we have*

$$\mathbf{P}' \cong \mathbf{N_G(H)}/\mathbf{C_G(H)}, \tag{29}$$

the order of which is equal to q *(Eq. 24).*

This theorem is well-known; see Theorem 3.5 in Chapter 1 of Ref. 25.

Moreover, the group $\mathbf{P}'$ (order q) is a subgroup of $\mathbf{P}$ (order $\varphi(|\mathbf{H}|)$) that is regarded as Aut $\mathbf{H}$ (Eq. 4 and Theorem 7 in Appendix A). Theorem 2 shows that the maturity discriminant q[23] is equal to the order of the group $\mathbf{P}'$.

Since $\mathbf{P}'$ and its supergroup $\mathbf{P}$ (Eq. 4 and Theorem 7 in Appendix A) act on the same cyclic group $\mathbf{H}$ as an ordered set, the orbit governed by $\mathbf{P}(/\{I\})$ discribed in Theorem 8 (Appendix A) can be further partitioned into $\varphi(|\mathbf{H}|)/q$ of suborbits if the subgroup $\mathbf{P}'$ is taken into consideration. When $\mathbf{P}'$ acts on $\mathbf{H}$, or in other words, when t_i runs over $\mathbf{B}'$, the element $t_i^{-1}\tilde{h}t_i$ creates a conjugacy class. Hence, each of the suborbits is identical with the conjugacy class $\mathbf{K}_{ij} \cap \mathbf{H}$. Note that the number $\varphi(|\mathbf{H}|)/q$ is equal to the number t of such conjugacy classes as given by

$$t = \frac{\varphi(|\mathbf{H}|)|\mathbf{C_G(H)}|}{|\mathbf{N_G(H)}|}. \tag{30}$$

This discussion is summarized as a theorem.

Theorem 3. *Let* $\mathbf{H}$ *be a cyclic subgroup of* $\mathbf{G}$. The dominant class $\mathbf{K}_i \cap \mathbf{H}$ in the normalizer $\mathbf{N_G(H)}$ is identical with the orbit by $\mathbf{P}$ described in Theorem 8. Each conjugacy class $\mathbf{K}_{ij} \cap \mathbf{H}$ is identical to the suborbit generated by the action of $\mathbf{P}'$ of Theorem 2.

Let us consider a conjugacy class generated as a suborbit (Theorem 3):

$$\{t_1^{-1}\tilde{h}t_1, t_2^{-1}\tilde{h}t_2, \cdots, t_q^{-1}\tilde{h}t_q\} = \{\tilde{h}^{r_1}, \tilde{h}^{r_2}, \cdots, \tilde{h}^{r_q}\}, \tag{31}$$

which produces a permutation:

$$P_{\tilde{h}}^{t_i} = \begin{pmatrix} t_1^{-1}\tilde{h}t_1 & t_2^{-1}\tilde{h}t_2 & \cdots & t_q^{-1}ht_q \\ t_i^{-1}t_1^{-1}\tilde{h}t_1t_i & t_i^{-1}t_2^{-1}\tilde{h}t_2t_i & \cdots & t_i^{-1}t_q^{-1}\tilde{h}t_qt_i \end{pmatrix} \tag{32}$$

for $t_i \in \mathbf{B}'$. Obviously, this permutation is a fraction of the permutation P_{t_i} and generates the suborbit concerning the conjugacy class. Since we have $t_it_j \in \mathbf{N}$, we can place $t_it_j = ct \in \mathbf{C}t$. Hence, Eq. 26 gives a lemma:

Lemma 1. *The permutation* $P_{\tilde{h}}^{t_i}$ *(Eq. 32) produces every element of the conjugacy class in a respective column of the permutations when* t_i *runs over* $\mathbf{B}'$.

It should be noted here that the orbit by $\mathbf{P}$ is associated with the set $\mathbf{Z}_n^*$ of integers coprime to n ($= |\mathbf{H}|$):

$$\mathbf{Z}_n^* = \{s | (s, n) = 1\}, \tag{33}$$

where we assume $s < n$ for simplicity's sake.

On the other hand, the suborbits by $\mathbf{P}'$ result in a further partition of the set of the integers. Let us select a representative $\tilde{h}_j$ from the conjugacy class $\mathbf{K}_{ij} \cap \mathbf{H}$ ($j = 1, 2, \cdots, t$). Then, we use the set of such representatives:

$$\mathbf{E} = \{\tilde{h}_j | j = 1, 2, \cdots, t\}, \tag{34}$$

where t is given by Eq. 30. Thereby, the subsets derived by the action $\mathbf{P}'$ are represented as follows:

$$\mathbf{Z}_{\tilde{h}} = \{r | \tilde{h}^r = t^{-1}\tilde{h}t, \forall t \in \mathbf{B}'\}, \tag{35}$$

for $\tilde{h} \in \mathbf{E}$, where we have $(r, n) = 1$ and we assume $s < n$ for simplicity's sake. Note that $|Z_{\tilde{h}}| = |\mathbf{B}'| = q$. Since the subsets $\mathbf{Z}_{\tilde{h}}$ are disjoint, we have

$$\mathbf{Z}_n^* = \sum_{\tilde{h} \in \mathbf{E}} \mathbf{Z}_{\tilde{h}} \tag{36}$$

Suppose that $\tilde{h}$ and $\tilde{h}^s$ are not conjugate so that we have $\tilde{h}^s \in \mathbf{E}$. From the condition $\tilde{h}^r = t^{-1}\tilde{h}t$ for Eq. 35, we obtain

$\tilde{h}^{rs} = t^{-1}\tilde{h}^{s}t$. Since the assumption indicates that $\tilde{h}$ and $t^{-1}\tilde{h}^{s}t$ are not conjugate, we have $rs \notin \mathbf{Z}_{\tilde{h}}$. Hence, we obtain

$$\mathbf{Z}_{\tilde{h}^s} = \{r' | \tilde{h}^{r'} = t^{-1}\tilde{h}^{s}t, \forall t \in \mathbf{B}'\}, \tag{37}$$

where we have $r' \equiv rs \pmod{n}$. Note that the integer r' runs so as to correspond to r in one-to-one fashion. This means that the elements of $\mathbf{Z}_{\tilde{h}^s}$ (Eq. 37) correspond to those of $\mathbf{Z}_{\tilde{h}}$ (Eq. 35) in one-to-one fashion. Since element $\tilde{h}^s$ runs over $\mathbf{E}$, the sets of the disjoint union (summation) appearing on the right-hand side of Eq. 36 can be transformed to each other.

1.5 Inherent Automorphisms and Inner Automorphisms. 1.5.1 Selecting Representatives of Cosets. The symbols **H**, **C**, **N**, and **G** are used with the same meanings as above. Let $\mathbf{H}x$ and $\mathbf{H}x'$ be distinct cosets ($\mathbf{H}x \neq \mathbf{H}x'$) that are contained in a coset decomposition of **G** by a cyclic subgroup **H**, as shown in Eq. 5. For $\tilde{t} \in \mathbf{N}$, we consider a similarity transformation:

$$\tilde{t}^{-1}\mathbf{H}x'\tilde{t} = (\tilde{t}^{-1}\mathbf{H}\tilde{t})(\tilde{t}^{-1}x'\tilde{t}) = \mathbf{H}^{\tilde{h}}(\tilde{t}^{-1}x'\tilde{t}). \tag{38}$$

Suppose that the coset represented by Eq. 38 (as unordered sets) is equal to $\mathbf{H}x$, but that their representatives are not identical with each other. This means that these cosets satisfy

$$\mathbf{H}x = \tilde{t}^{-1}\mathbf{H}x'\tilde{t} \tag{39}$$

$$x \neq \tilde{t}^{-1}x'\tilde{t} \tag{40}$$

for $\tilde{t} \in \mathbf{N}$. Because of Eq. 39, we have $\exists \tilde{h}^{a}x'$ satisfying

$$x = \tilde{t}^{-1}\tilde{h}^{a}x'\tilde{t}, \tag{41}$$

where a is an integer. The coset (Eq. 38) can be written as a concrete form:

$$\tilde{t}^{-1}\mathbf{H}x'\tilde{t} = \{\tilde{t}^{-1}\tilde{h}x'\tilde{t}, \tilde{t}^{-1}\tilde{h}^{2}x'\tilde{t}, \cdots, \tilde{t}^{-1}\tilde{h}^{r}x'\tilde{t}(=x), \cdots, \tilde{t}^{-1}x'\tilde{t}\}. \tag{42}$$

This coset is rearranged so as to place $\tilde{t}^{-1}\tilde{h}^{a+1}x'\tilde{t}$ as a top element (or we take account of $(\tilde{t}^{-1}\tilde{h}^{a}\tilde{t})(\tilde{t}^{-1}\mathbf{H}x'\tilde{t}) = \tilde{t}^{-1}\tilde{h}^{a}\mathbf{H}x'\tilde{t}$). Thereby, we obtain

$$\begin{aligned}\tilde{t}^{-1}\tilde{h}^{a}\mathbf{H}x'\tilde{t} &= \{\tilde{t}^{-1}\tilde{h}^{a+1}x'\tilde{t}, \tilde{t}^{-1}\tilde{h}^{a+2}x'\tilde{t}, \cdots, \tilde{t}^{-1}\tilde{h}^{a}x'\tilde{t}(=x)\} \\ &= \{\tilde{t}^{-1}\tilde{h}\tilde{t}x, \tilde{t}^{-1}\tilde{h}^{2}\tilde{t}x, \cdots, x\} \\ &= \mathbf{H}\tilde{t}x.\end{aligned} \tag{43}$$

Hence, we are able to select x (Eq. 41) as a representative in place of $\tilde{t}^{-1}x'\tilde{t}$. The representative ($x'$) of the original coset $\mathbf{H}x'$ is now changed into $x'' = \tilde{h}^{a}x'$ so as to give $\mathbf{H}x' = \mathbf{H}x''$. Then we have

$$\tilde{t}^{-1}x''\tilde{t} = \tilde{t}^{-1}\tilde{h}^{a}x'\tilde{t} = x. \tag{44}$$

This means that the element x'' is conjugate to x. The discussions described above are summarized to give the following Lemma:

Lemma 2. *Let* $\mathbf{H}x$ *and* $\mathbf{H}x'$ *be distinct cosets* ($\mathbf{H}x \neq \mathbf{H}x'$) *that are contained in a coset decomposition of* **G** *by a cyclic subgroup* **H**. *Let* **N** *be the normalizer of* **H** *within* **G**. *The representative x and x′ can be selected to be conjugate with each other*, i.e., $x = \tilde{t}^{-1}x'\tilde{t}$, *so that we have*

$$\tilde{t}^{-1}\mathbf{H}x'\tilde{t} = (\tilde{t}^{-1}\mathbf{H}\tilde{t})(\tilde{t}^{-1}x'\tilde{t}) = \mathbf{H}^{\tilde{t}}(\tilde{t}^{-1}x'\tilde{t}) = \mathbf{H}^{\tilde{t}}x \tag{45}$$

for $\tilde{t} \in \mathbf{N}$.

Let us next consider a special case of $\mathbf{H}x = \mathbf{H}x'$ described in the preceding discussion. Thus, suppose that

$$\mathbf{H}x = \tilde{t}^{-1}\mathbf{H}x\tilde{t} \tag{46}$$

$$x \neq \tilde{t}^{-1}x\tilde{t} \tag{47}$$

for $\tilde{t} \in \mathbf{N}$. Because of Eq. 46, we have $\exists \tilde{h}^{a}x$ satisfying

$$x = \tilde{t}^{-1}\tilde{h}^{a}x\tilde{t}, \tag{48}$$

where a is an integer. The coset appearing in the right-hand side of Eq. 46 can be written as a concrete form:

$$\begin{aligned}\tilde{t}^{-1}\mathbf{H}x\tilde{t} = \{&\tilde{t}^{-1}\tilde{h}x\tilde{t}, \tilde{t}^{-1}\tilde{h}^{2}x\tilde{t}, \cdots, \tilde{t}^{-1}\tilde{h}^{a}x\tilde{t}(=x), \tilde{t}^{-1}\tilde{h}^{a+1}x\tilde{t}(=\tilde{h}x), \\ &\tilde{t}^{-1}\tilde{h}^{a+2}x\tilde{t}(=\tilde{h}^{2}x), \cdots, \tilde{t}^{-1}x\tilde{t}\}.\end{aligned} \tag{49}$$

This coset is rearranged so as to place $\tilde{t}^{-1}\tilde{h}^{a+1}x\tilde{t}$ as a top element (or we take account of $(\tilde{t}^{-1}\tilde{h}^{a}\tilde{t})(\tilde{t}^{-1}\mathbf{H}x\tilde{t}) = \tilde{t}^{-1}\tilde{h}^{a}\mathbf{H}x\tilde{t}$). Thereby, we obtain

$$\begin{aligned}\tilde{t}^{-1}\tilde{h}^{a}\mathbf{H}x\tilde{t} &= \{\tilde{t}^{-1}\tilde{h}^{a+1}x\tilde{t}, \tilde{t}^{-1}\tilde{h}^{a+2}x\tilde{t}, \cdots, \tilde{t}^{-1}\tilde{h}^{a}x\tilde{t}(=x)\} \\ &= \{\tilde{t}^{-1}\tilde{h}\tilde{t}x, \tilde{t}^{-1}\tilde{h}^{2}\tilde{t}x, \cdots, x\} \\ &= \mathbf{H}^{\tilde{t}}x.\end{aligned} \tag{50}$$

Hence, we are able to obtain $\mathbf{H}x$ and $\mathbf{H}^{\tilde{t}}x$ with a common representative x. As a result, Lemma 2 is modified to treat the other case.

Lemma 3. *Let* $\mathbf{H}x$ *and* $\tilde{t}^{-1}\mathbf{H}x\tilde{t}$ *satisfy Eqs. 46 and 47 for* $\tilde{t} \in \mathbf{N}$. *Then, the elements of the coset* $\tilde{t}^{-1}\mathbf{H}x\tilde{t}$ *can be permuted to give* $\mathbf{H}^{\tilde{t}}x$ *that has a common representative x with the coset* $\mathbf{H}x$.

1.5.2 Inner Portion of the Inherent Automorphism Group. By starting from Eq. 5, we have another coset decomposition:

$$\begin{aligned}\mathbf{G} &= \tilde{t}^{-1}\mathbf{H}\tilde{t} \cdots + \tilde{t}^{-1}\mathbf{H}x\tilde{t} + \cdots + \tilde{t}^{-1}\mathbf{H}x'\tilde{t} + \cdots \\ &= \tilde{t}^{-1}\mathbf{H}\tilde{t} + \cdots + (\tilde{t}^{-1}\mathbf{H}\tilde{t})(\tilde{t}^{-1}x\tilde{t}) + \cdots + (\tilde{t}^{-1}\mathbf{H}\tilde{t})(\tilde{t}^{-1}x'\tilde{t}) + \cdots \\ &= \mathbf{H}^{\tilde{t}} + \cdots + \mathbf{H}^{\tilde{t}}(\tilde{t}^{-1}x\tilde{t}) + \cdots + \mathbf{H}^{\tilde{t}}(\tilde{t}^{-1}x'\tilde{t}) + \cdots,\end{aligned} \tag{51}$$

where we perform a similarity transformation concerning an element $\tilde{t}$ of the normalizer **N** of **H** and we use $\tilde{t}^{-1}\mathbf{G}\tilde{t} = \mathbf{G}$. Note that $\mathbf{H}^{\tilde{t}}$ $(= \tilde{t}^{-1}\mathbf{H}\tilde{t})$ is identical with **H** as an unordered set, though we can differentiate them as ordered sets if necessary. If we regard **H** and $\mathbf{H}^{\tilde{t}}$ as ordered sets, the transformation from Eq. 5 to Eq. 51 is an inner automorphism.

If we identify **H** with $\mathbf{H}^{\tilde{t}}$ as unordered sets, Eqs. 5 and 51 provide a permutation represented by

$$\tilde{P}_{\tilde{t}} = \begin{pmatrix} \mathbf{H} & \cdots & \mathbf{H}x & \cdots & \mathbf{H}x' & \cdots \\ \mathbf{H}^{\tilde{t}} & \cdots & \mathbf{H}^{\tilde{t}}(\tilde{t}^{-1}x\tilde{t}) & \cdots & \mathbf{H}^{\tilde{t}}(\tilde{t}^{-1}x'\tilde{t}) & \cdots \end{pmatrix}. \tag{52}$$

Since Lemma 2 (Eq. 45) indicates

$$\mathbf{H}x' = \tilde{t}^{-1}(\mathbf{H}\tilde{t}x\tilde{t}^{-1})\tilde{t} = \tilde{t}^{-1}\mathbf{H}\tilde{t}x = \mathbf{H}^{\tilde{t}}x, \tag{53}$$

we have

$$\tilde{P}_{\tilde{t}} = \begin{pmatrix} \mathbf{H} & \cdots & \mathbf{H}x & \cdots & \mathbf{H}\tilde{t}x\tilde{t}^{-1} & \cdots \\ \mathbf{H}^{\tilde{t}} & \cdots & \mathbf{H}^{\tilde{t}}(\tilde{t}^{-1}x\tilde{t}) & \cdots & \mathbf{H}^{\tilde{t}}x & \cdots \end{pmatrix} \quad (54)$$

An invariant coset on the action of the permutation **P** is controlled by Lemma 3. When $\tilde{t}$ runs over **N**, the permutations of 54 construct a set represented by

$$\tilde{\mathbf{P}} = \{\tilde{P}_{\tilde{t}} | \tilde{t} \in \mathbf{N}\}, \quad (55)$$

where the mapping $\tilde{\mathbf{P}} \to \mathbf{N}$ is a homomorphism.

Assume that $x = \tilde{t}^{-1}x\tilde{t}$ ($\forall x \in \mathbf{G}$). Then the $\tilde{t}$ is contained in the center $\mathbf{Z}(\mathbf{G})$ of **G**. The $\tilde{t}$ keeps all of the cosets in Eq. 52 invariant; in other words, the resulting permutation $\tilde{P}_{\tilde{t}}$ is equal to an identity. Hence, the kernel of the homomorphism $\tilde{\mathbf{P}} \to \mathbf{N}$ is the center $\mathbf{Z}(\mathbf{G})$, giving an isomorphism:

$$\tilde{\mathbf{P}} \cong \mathbf{N}/\mathbf{Z}(\mathbf{G}). \quad (56)$$

An inner automorphism $\tilde{P}^{\tilde{t}}$ is obtained from Eq. 54 by taking explicit account of the elements of **H** and those of $\mathbf{H}^{\tilde{t}}$. By paying simple attention to the coset $\mathbf{H}^{\tilde{t}}$, such $\tilde{P}^{\tilde{t}}$ can be written to be

$$\tilde{P}^{\tilde{t}} = \begin{pmatrix} \cdots; & \tilde{h}x' & \tilde{h}^2x' & \cdots; \cdots \\ \cdots; & (\tilde{t}^{-1}\tilde{h}\tilde{t})(\tilde{t}^{-1}x'\tilde{t}) & (\tilde{t}^{-1}\tilde{h}^2\tilde{t})(\tilde{t}^{-1}x'\tilde{t}) & \cdots; \cdots \end{pmatrix} \quad (57)$$

When $\tilde{t}$ runs over the normalizer **N**, the inner automorphisms $\tilde{P}^{\tilde{t}}$ constructs an inner autophorphism group.

Let us now examine the coset $\mathbf{H}^{\tilde{t}}x'$ appearing in $\tilde{P}^{\tilde{t}}$ (Eq. 57). Lemma 2 gives a permutation concerning the elements of $\mathbf{H}x'$:

$$\begin{pmatrix} \mathbf{H}x' \\ \mathbf{H}^{\tilde{t}}(\tilde{t}^{-1}x'\tilde{t}) \end{pmatrix} = \begin{pmatrix} \mathbf{H}x' \\ \mathbf{H}^{\tilde{t}}x' \end{pmatrix} \begin{pmatrix} \mathbf{H}^{\tilde{t}}x' \\ H^{\tilde{t}}(\tilde{t}^{-1}x'\tilde{t}) \end{pmatrix} = \begin{pmatrix} \mathbf{H}x' \\ \mathbf{H}^{\tilde{t}}x' \end{pmatrix} \begin{pmatrix} \mathbf{H}^{\tilde{t}}x' \\ \mathbf{H}^{\tilde{t}}x \end{pmatrix}, \quad (58)$$

where x' runs to construct the inner automorphism. Lemma 3 gives a similar result. The second permutation in the last side of Eq. 58 corresponds to $\tilde{P}_{\tilde{t}}$ described in Eq. 52.

On the other hand, the first permutation in the last side of Eq. 58 is written in a concrete form to give

$$L_{x'}^{\tilde{t}} = \begin{pmatrix} \mathbf{H}x' \\ \mathbf{H}^{\tilde{t}}x' \end{pmatrix} = \begin{pmatrix} \tilde{h} & \tilde{h}^2 & \cdots & \tilde{h}^n \\ \tilde{t}^{-1}\tilde{h}\tilde{t} & \tilde{t}^{-1}\tilde{h}^2\tilde{t} & \cdots & \tilde{t}^{-1}\tilde{h}^n\tilde{t} \end{pmatrix} \approx \begin{pmatrix} \mathbf{H} \\ \mathbf{H}^{\tilde{t}} \end{pmatrix}, \quad (59)$$

the last of which is the same as P_{t_i} of Eq. 27, where we place $\tilde{t} = t_i$. When x' runs to cover all of the cosets, we obtain

$$L^{\tilde{t}} = \begin{pmatrix} \mathbf{H} & \cdots & \mathbf{H}x' & \cdots \\ \mathbf{H}^{\tilde{t}} & \cdots & \mathbf{H}^{\tilde{t}}x' & \cdots \end{pmatrix} = \begin{pmatrix} \tilde{h} & \tilde{h}^2 & \cdots; & \cdots; & \tilde{h}x' & \tilde{h}^2x' & \cdots; & \cdots \\ \tilde{t}^{-1}\tilde{h}\tilde{t} & \tilde{t}^{-1}\tilde{h}^2\tilde{t} & \cdots; & \cdots; & \tilde{t}^{-1}\tilde{h}\tilde{t}x' & \tilde{t}^{-1}\tilde{h}^2\tilde{t}x' & \cdots; & \cdots \end{pmatrix}. \quad (60)$$

In the light of Eq. 60, the correspondence $\tilde{h} \to \tilde{t}^{-1}\tilde{h}\tilde{t}$ provides a permutation $L^{\tilde{t}}$ by starting from the relationship described in Eq. 59 in a similar way to that described in Def. 1.

The above discussion shows that the $L^{\tilde{t}}$ is an automorphism of **G**.

$$\mathbf{L}' = \{L^{\tilde{t}} | \tilde{t} \in \mathbf{N}\}, \quad (61)$$

Theorem 2 along with Eq. 59 shows that

$$\mathbf{L}' \cong \mathbf{P}' \cong \mathbf{N}/\mathbf{C}. \quad (62)$$

Moreover, by virtue of Eq. 27, the automorphism $L^{\tilde{t}}$ derived from the correspondence $\tilde{h} \to \tilde{t}^{-1}\tilde{h}\tilde{t}$ is involved in the **H**-inherent automorphism group of **G**. We define 'the inner portion of the inherent automorphism group' as follows:

Definition 2. (Inner Portion of the Inherent Automorphism Group) *By using the transversal* $\mathbf{B}'$ *(Eq. 23), we have the set of all permutations* $L^{\tilde{t}}$ *represented by*

$$\mathrm{Por}_{\mathbf{H}}(\mathbf{G}) = \mathbf{L}' = \{L^{\tilde{t}} | \tilde{t} \in \mathbf{B}'\}, \quad (63)$$

which is a group isomorphic to $\mathbf{P}'$ *(Eq. 63). Obviously, the group* $\mathbf{L}'$ *(order q) is a subgroup of* $\mathrm{Inh}_{\mathbf{H}}(\mathbf{G})$ *(order* $\varphi(|\mathbf{H}|)$*). We call the group* $\mathbf{L}'$ *the 'inner portion of the inherent automorphism group' of* **G**, *which is designated by the symbol* $\mathrm{Por}_{\mathbf{H}}(\mathbf{G})$.

It is worthwhile to mention the relationship between $L^{(s)}$ (Def. 1) and $L^{\tilde{t}}$ (Def. 2). Thus, the concrete form of $L^{(s)}$ can be written as follows:

$$L^{(s)} = \begin{pmatrix} \mathbf{H} & \cdots & \mathbf{H}x' & \cdots \\ \mathbf{H}^{(s)} & \cdots & \mathbf{H}^{(s)}x' & \cdots \end{pmatrix} = \begin{pmatrix} \tilde{h} & \tilde{h}^2 & \cdots; & \cdots; & \tilde{h}x' & \tilde{h}^2x' & \cdots; & \cdots \\ \tilde{h}^s & \tilde{h}^{2s} & \cdots; & \cdots; & \tilde{h}^sx' & \tilde{h}^{2s}x' & \cdots; & \cdots \end{pmatrix} \quad (64)$$

By comparing this with Eq. 60, these two permutations can be identified if the elements $\tilde{h}$ and $\tilde{h}^s$ satisfy $\tilde{t}^{-1}\tilde{h}\tilde{t} = \tilde{h}^s$.

2 Maturity of Irreducible Representations

2.1 Inherent Set of Irreducible Characters. Let $\mathbf{H}x$ be a coset of a group **G** by a cyclic subgroup **H**, where the representative x is selected to satisfy Eq. 5. Select an irreducible character $\theta(\tilde{h}^a x)$ as a standard, each element of which is a trace of an irreducible representation $\boldsymbol{\Theta}(\tilde{h}^a x)$ for $\tilde{h}^a x \in \mathbf{H}x$ ($a = 1, 2, \cdots, n$).

The correspondence $\tilde{h} \to \tilde{h}^s$ gives another character:

$$\theta^{(s)}((\tilde{h}^s)^a x) = \theta(\tilde{h}^a x), \quad (65)$$

where the integer s is coprime to n, i.e. $(s,n) = 1$. Since the correspondence $\tilde{h} \to \tilde{h}^s$ generates an inherent automorphism of **G**, the resulting representation (Eq. 65) is concluded to be an irreducible character (IC). When the integer a s runs so as to satisfy $(s,n) = 1$, it gives the set of irreducible characters (ICs) $\theta^{(s)}$, which is called the *inherent set of ICs* concerning **H**:

$$\boldsymbol{\Theta} = \{\theta^{(s)} | (s,n) = 1\} = \{\theta^{(s)} | s \in \mathbf{Z}_n^*\} \quad (66)$$

Although the inherent set of ICs involves $\varphi(|\mathbf{H}|)$ of characters at most, it may be integrated into one IC to give another extreme case, which is regarded as a matured one. It should be noted that the former case having $\varphi(|\mathbf{H}|)$ of ICs corresponds to a fully unmatured case concerning the cyclic subgroup $\mathbf{H}$. Obviously, a case in which $\mathbf{G} = \mathbf{H}$ is such an extreme case, where the inherent set of ICs is identical with the set of primitive irreducible characters described elsewhere.[23)]

The character $\theta^{(s)}$ (Eq. 65) is related to the permutation $L^{(s)}$ (Eq. 64). Thus, the right-hand side of Eq. 65 is concerned with the starting set of elements in $L^{(s)}$, while the left-hand side of Eq. 65 is concerned with the resulting set of elements in $L^{(s)}$. In order to compare the character $\theta^{(s)}$ with the original one θ, the elements for $\theta^{(s)}$ should be rearranged to the original alignment. This rearrangement can be replaced by another rearrangement based on $\tilde{P}^{\tilde{t}}$ (Eq. 64), since the resulting alignment in each coset of $L^{(s)}$ is the same as that in the corresponding coset of $\tilde{P}^{\tilde{t}}$, if we have $\tilde{h}^s = \tilde{t}^{-1}\tilde{h}\tilde{t}$. It follows that

$$\theta^{(s)}(\tilde{t}^{-1}\tilde{h}^a\tilde{t}x) = \theta^{(s)}((\tilde{h}^s)^a x) = \theta(\tilde{h}^a x), \tag{67}$$

for $a = 1,2,\cdots n$. This situation of rearrangements is illustrated as follows:

$$\begin{array}{ccccc}
\cdots & \theta^{(s)}\left(\left(\tilde{h}^s\right)^a x\right) & \cdots & \cdots & \cdots \\
\cdots & \mathbf{H}^{(s)}x & \cdots & \cdots & \cdots \\
 & \uparrow L^{(s)} & \text{(see Eq. 64)} & & \\
\cdots & \mathbf{H}x & \cdots & \mathbf{H}\tilde{t}x\tilde{t}^{-1} & \cdots \\
\cdots & \theta\left(\tilde{h}^a x\right) & \cdots & \theta\left(\tilde{h}^a\tilde{t}x\tilde{t}^{-1}\right) & \cdots \\
 & \downarrow \tilde{P}^{\tilde{t}} & \text{(see Eqs. 57 and 54)} & \downarrow \tilde{P}^{\tilde{t}} & \\
\cdots & \mathbf{H}^{\tilde{t}}\tilde{t}^{-1}x\tilde{t} & \cdots & \mathbf{H}^{\tilde{t}}x & \cdots \\
\cdots & \theta^{(s)}\left(\tilde{t}^{-1}\tilde{h}^a x\tilde{t}\right) & \cdots & \theta^{(s)}\left(\tilde{t}^{-1}\tilde{h}^a\tilde{t}x\right) & \cdots
\end{array}$$

The position of $\mathbf{H}^{\tilde{t}}x$ in the resulting alignment depends on $\tilde{t}$ through the starting $\mathbf{H}\tilde{t}x\tilde{t}^{-1}$. However, we are able to collect $\theta^{(s)}(\tilde{t}^{-1}\tilde{h}^a\tilde{t}x)$ ($= \theta(\tilde{h}^a x)$ as shown in Eq. 67) corresponding to $\mathbf{H}^{\tilde{t}}x$, where $\tilde{t}$ runs over $\mathbf{B}'$ (x is tentatively fixed).

The total result of rearrangements given by Eq. 67 means that the rearrangement within the coset $\mathbf{H}x$ is replaced by the rearrangement between the cosets of $\mathbf{G}$. Thereby, the character by Eq. 67 is given by the inderect method via $L^{(s)}$ and $\tilde{P}^{\tilde{t}}$. On the other hand, such a character can be given more directly if we use $L^{\tilde{t}}$ (Eq. 60), where we are able to limit our discussion within the coset $\mathbf{H}x$. This shall be done in the following paragraphs.

Let us here consider the correspondence $\tilde{h} \to \tilde{t}^{-1}\tilde{h}\tilde{t}$ (where $\tilde{h} \in E$) to create another type of representations:

$$\theta_{\tilde{t}}\left(\left(\tilde{t}^{-1}\tilde{h}^a\tilde{t}\right)x\right) = \theta^{(r)}((\tilde{h}^r)^a x) = \theta\left(\tilde{h}^a x\right) \tag{68}$$

for $a = 1,2,\cdots,n$, where $\tilde{t}$ runs over the normalizer $\mathbf{N_G}(\mathbf{H})$ or equivalently over the transversal $\mathbf{B}'$ (Eq. 23). Since $\tilde{t}^{-1}\tilde{h}\tilde{t}$ is an element of $\mathbf{H}$, we put $\tilde{t}^{-1}\tilde{h}\tilde{t} = \tilde{h}^r$, where the integers r satisfy $(r,n) = 1$. Since the correspondence $\tilde{h} \to \tilde{t}^{-1}\tilde{h}\tilde{t}$ (i.e. $\tilde{h} \to \tilde{h}^r$) generates an inherent automorphism of $\mathbf{G}$, the resulting representation (Eq. 66) is concluded to be an irreducible character (IC). When $\tilde{t}$ runs over $\mathbf{B}'$, the integer r runs over $\mathbf{Z}_{\tilde{h}}$ (Eq. 35). Hence, we obtain the set of such ICs,

$$\boldsymbol{\Theta}'_{\tilde{h}} = \{\theta^{(r)} | r \in \mathbf{Z}_{\tilde{h}}\}, \tag{69}$$

for $\tilde{h} \in \mathbf{E}$ (Eq. 34). These ICs are involved in the inherent set of ICs $\boldsymbol{\Theta}$ (Eq. 66).

Let us consider an element $t^{-1}\tilde{h}xt$ conjugate to $\tilde{h}x \in \mathbf{H}x$, where $\tilde{h}$ is selected from $\mathbf{E}$ (Eq. 34). Since we have

$$t^{-1}\tilde{h}xt = (t^{-1}\tilde{h}t)(t^{-1}xt) \in \mathbf{H}(t^{-1}xt), \tag{70}$$

the correspondence $\tilde{h} \to \tilde{t}^{-1}\tilde{h}\tilde{t}$ gives

$$[t^{-1}(\tilde{t}^{-1}\tilde{h}\tilde{t})t](t^{-1}xt) = t^{-1}(\tilde{t}^{-1}\tilde{h}\tilde{t})xt, \tag{71}$$

which is conjugated to the element $(\tilde{t}^{-1}\tilde{h}\tilde{t})x$ derived from $\tilde{h}x$. This means that the correspondence $\tilde{h} \to \tilde{t}^{-1}\tilde{h}\tilde{t}$ (and equivalently $\mathrm{Por}_{\mathbf{H}}(\mathbf{G})$) keeps the conjugacy relationship invariant. Hence, the correspondence gives the same IC as the original one (Eq. 66). When the element $\tilde{t}$ runs so that the corresponding integer r is involved in $\mathbf{Z}_{\tilde{h}}$ (the element $\tilde{h}$ is tentatively fixed), the resulting characters $\theta^{(r)}$ ($\in \boldsymbol{\Theta}'_{\tilde{h}}$) are equal to each other.

Lemma 4. *The set $\boldsymbol{\Theta}'_{\tilde{h}}$ (Eq. 69) consists of one irreducible character (multiplicity = q), when $\tilde{h}$ is fixed.*

Let us again consider the element $t^{-1}\tilde{h}xt$ conjugate to $\tilde{h}x \in \mathbf{H}x$, as shown in Eq. 70. Then, an element $\tilde{h}^s x \in \mathbf{H}^{(s)}x$ satisfies

$$t^{-1}\tilde{h}^s xt = (t^{-1}\tilde{h}^s t)(t^{-1}xt) = (t^{-1}\tilde{h}t)^s(t^{-1}xt) \in \mathbf{H}^{(s)}(t^{-1}xt) \tag{72}$$

for an integer s satisfying $(s,n) = 1$. Note that $\tilde{h}x$ and $\tilde{h}^s x$ are not always conjugate to each other. Equation 72 shows that the correspondence $\tilde{h} \to \tilde{h}^s$ keeps the conjugacy relationship invariant. This is summarized to give a lemma.

Lemma 5. *An inherent automorphism (Def. 1) keeps conjugacy relationships invariant.*

Let us now consider the restriction of $\mathbf{G}$ into its cyclic subgroup $\mathbf{H}$. Then, the character $\theta^{(r)}$ of $\boldsymbol{\Theta}'_{\tilde{h}}$ (Eq. 69) is restricted to be

$$(\theta_{\tilde{t}})_{\downarrow\mathbf{H}} = (\theta^{(r)})_{\downarrow\mathbf{H}} = \sum_{r_i \in \mathbf{Z}_{\tilde{h}}} m_{r_i}\Gamma_{\tilde{h}^{r_i}} \tag{73}$$

by means of ICs $\Gamma_{\tilde{h}^{r_i}}$ of $\mathbf{H}$, where $\tilde{h}$ ($\in \mathbf{E}$) given by Eq. 34 is tentatively fixed. Note that the symbol $\Gamma_{\tilde{h}^{r_i}}$ is used for designating a representation and its character, since the $\Gamma_{\tilde{h}^{r_i}}$ is a representation of one degree. Because the restricted IC $(\theta^{(r)})_{\downarrow\mathbf{H}}$ is invariant for any $t \in \mathbf{B}'$, or equivalently for any $r_i \in \mathbf{Z}_{\tilde{h}}$, all of the multiplicities m_{r_i} are equal to each other, i.e. $m_{r_i} = m$. As a result, Eq. 73 can be rewritten as

$$(\theta^{(r)})_{\downarrow\mathbf{H}} = m\sum_{r_i \in \mathbf{Z}_{\tilde{h}}} \Gamma_{\tilde{h}^{r_i}}. \tag{74}$$

This result means that an appropriate similarity transformation gives a representation $\Theta^{(r)}$ consisting of diagonal matrices for its $\mathbf{H}$ part:

$$\Theta^{(r)}(\tilde{h}^a) = [\omega^{ar_1}, \omega^{ar_1}, \cdots, \omega^{ar_2}, \omega^{ar_2}, \cdots, \omega^{ar_q}, \omega^{ar_q}, \cdots], \tag{75}$$

for $a = 1,2,\cdots,n$, where only the diagonal elements are shown for simplicity's sake and all of the off-diagonal elements are equal to zero. The symbol ω designates a primitive nth root of 1 and the integers $r_1, r_2, \cdots$ are the elements of $\mathbf{Z}_{\tilde{h}}$. The corresponding character is the sum of the diagonal elements of Eq. 75, i.e.,

$$\theta^{(r)}(\tilde{h}^a) = \underbrace{\omega^{ar_1} + \omega^{ar_1} + \cdots}_{m} + \underbrace{\omega^{ar_2} + \omega^{ar_2} + \cdots}_{m} + \underbrace{\omega^{ar_q} + \omega^{ar_q} + \cdots}_{m}$$
$$= m(\omega^{ar_1} + \omega^{ar^{ar_2}} + \cdots + \omega^{ar_q}). \quad (76)$$

In the light of Lemma 4, we assume a matrix for $\theta^{(r)}(x)$:

$$\theta^{(r)}(x) = [x_1, x_2, \cdots, x_1, x_2, \cdots, x_1, x_2, \cdots], \quad (77)$$

where only diagonal elements are shown and off-diagonal elements may be non-zero. Since the representation for the element $\tilde{h}^a x$ is $\theta^{(r)}(\tilde{h}^a x) = \theta^{(r)}(\tilde{h}^a)\theta^{(r)}(x)$, the corresponding character is calculated as follows:

$$\theta^{(r)}\left(\tilde{h}^a x\right) = (x_1 + x_2 + \cdots + x_m)(\omega^{ar_1} + \omega^{ar_2} + \cdots + \omega^{ar_q}), \quad (78)$$

where we take into consideration the fact that each $\theta^{(r)}(\tilde{h}^a)$ is a diagonal matrix. In summary, the character $\theta^{(r)}$ (Eqs. 76 and 78) is invariant if r runs over $\mathbf{Z}_{\tilde{h}}$, since r for $\theta^{(r)}$ is either element of $\mathbf{Z}_{\tilde{h}} = \{r_1, r_2, \cdots, r_q\}$.

When $\tilde{h}$ runs over $\mathbf{E}$, Eq. 74 creates t ($= \varphi(n)/q$) of ICs (cf. Lemma 4). These ICs are interconvertible in a similar way to that described for Eq. 37. It should be noted that the coeficient m is common even when $\tilde{h}$ runs over $\mathbf{E}$. The summation of ICs (Eq. 74) gives

$$\hat{\theta}_{\downarrow \mathbf{H}} = \sum_{\tilde{h} \in \mathbf{E}} (\theta^{(r)})_{\downarrow \mathbf{H}}$$
$$= \sum_{\tilde{h} \in \mathbf{E}} \left(m \sum_{r_i \in \mathbf{Z}_{\tilde{h}}} \Gamma_{\tilde{h}^{r_i}} \right) = m \sum_{\tilde{h} \in \mathbf{E}} \sum_{r_i \in \mathbf{Z}_{\tilde{h}}} \Gamma_{\tilde{h}^{r_i}}$$
$$= m \sum_{r_i \in \mathbf{Z}_n^*} \Gamma_{h^{r_i}} = m \hat{\gamma}_{n/n}. \quad (79)$$

The last side has been given in Theorem 1 of the preceding paper. That is to say, the values of $\tilde{\gamma}_{n/n}$ are equal if the corresponding elements are contained in the same dominant class. Hence, we have a lemma.

Lemma 6. *The terms corresponding to $\tilde{h}^s$ in Eq. 79 are equal to each other, where we have $(s,n) = 1$.*

If the sum $(x_1 + x_2 + \cdots + x_m)$ is a rational number, the sum $\sum \theta^{(r)}$ is concluded to be a rational character, as shown in Eq. 79. If the sum $(x_1 + x_2 + \cdots + x_m)$ is not a rational number, the process described in the preceding sections can be repeated to give a rational character. Hence, Lemma 6 describes a special case of Lemma 39.4 of Ref. 26.

2.2 Q-Conjugacy Character Tables. A result equivalent to Eq. 79 can be obtained by using Eq. 76.

$$\hat{\theta}(\tilde{h}^a) = \sum_{\tilde{h} \in \mathbf{E}} \theta^{(r)}(\tilde{h}^a)$$
$$= m \sum_{\tilde{h} \in \mathbf{E}} (\omega^{ar_1} + \omega^{ar_2} + \cdots + \omega^{ar_q})$$
$$= m \sum_{r_i \in \mathbf{Z}_n^*} \omega^{as} \quad (80)$$

which is the same value as described in Eq. 79. Suppose that $(a,n) = 1$. Then, Eq. 80 gives a more direct proof of Lemma 6.

Since Eq. 80 is not directly concerned with the restriction of $\mathbf{G}$ into its cyclic subgroup $\mathbf{H}$, the sums of irreducible characters for other parts of $\mathbf{G}$ can be obtained by using Eq. 78.

$$\hat{\theta}(\tilde{h}^a x) = \sum_{\tilde{h} \in \mathbf{E}} \theta^{(r)}(\tilde{h}^a x)$$
$$= (x_1 + x_1 + \cdots + x_m) \sum_{\tilde{h} \in \mathbf{E}} (\omega^{ar_1} + \omega^{ar_2} + \cdots + \omega^{ar_q}$$
$$= (x_1 + x_1 + \cdots + x_m) \sum_{s \in \mathbf{Z}_n^*} \omega^{as} \quad (81)$$

The reducible character $\hat{\theta}$ produced by Eqs. 80 and 81 is called a **Q**-*conjugacy character* in the present series of papers. This is obviously the non-redundant sum of the inherent set of irreducible characters (Eq. 66). The **Q**-conjugacy character corresponds to a reducible representation $\hat{\theta}$, which is called a **Q**-*conjugacy representation.*

The reducible character (Eqs. 80 and 81) contains t ($= \varphi(n)/q$) of ICs, since t is equal to $|\mathbf{E}|$, which represents the number of conjugacy classes in a dominant class. On the other hand, Lemma 6 shows that t of conjugacy classes has the same value as calculated in Eq. 79. In other words, **Q**-conjugacy characters along with markaracters are dominant-class functions, whereas the usual characters are class functions. Hence, the character table of **G** can be converted into a concise form in which t of the rows are summarized into one row (Eqs. 80 and 81) and t of the columns are summarized into one column (Lemma 6). Such a concise table is called a **Q**-*conjugacy character table* in the present series of papers. Suppose that a set of **Q**-conjugacy characters ($i = 1, 2, \cdots, s$) is represented by

$$\hat{\theta}_i = (\hat{\theta}_{i1}\, \hat{\theta}_{i2} \cdots \hat{\theta}_{is}), \quad (82)$$

where s is the number of dominant classes. Then, the corresponding **Q**-conjugacy character table is represented by

$$D = \begin{pmatrix} \hat{\theta}_{11} & \hat{\theta}_{12} & \cdots & \hat{\theta}_{1s} \\ \hat{\theta}_{21} & \hat{\theta}_{22} & \cdots & \hat{\theta}_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\theta}_{s1} & \hat{\theta}_{s2} & \cdots & \hat{\theta}_{ss} \end{pmatrix}. \quad (83)$$

Since each dominant class corresponds to a cyclic subgroup in one-to-one fashion, the following theorem is obtained easily.

Theorem 4. *The dimension of a* **Q**-*conjugacy character table is equal to that of the corresponding markaracter table described in Ref. 21. Thus, They are $s \times s$ matrices, where s is the number of dominant classes or equivalently the number of non-conjugate cyclic subgroups.*

Q-Conjugacy character tables prepared for every finite group are convenient for combinatorial enumeration of isomers. Such tables for several unmatured finite groups are collected in Appendix B. Note again that, for matured finite groups, **Q**-Conjugacy character tables are equivalent to character tables, since **Q**-conjugacy coincides with conjugacy in these cases.

2.3 Matured Representations and Matured Characters. In the discussion described above, a **Q**-conjugacy character is defined as a dominant-class function composed of an inherent set of ICs. Each of the ICs is inevitably associated with an irreducible representation. Hence, the **Q**-conjugacy character corresponds to a reducible representation made by the combination of the irreducible representations. The reducible representation is here called *a* **Q***-conjugacy representation.*

Let $\hat{\theta}_i$ $(i = 1,2,\cdots,s)$ be the set of **Q**-conjugacy representations and $\hat{\theta}_i$ $(i = 1,2,\cdots,s)$ the corresponding set **Q**-conjugacy characters, where s is equal to the number of non-conjugate subgroups of **G**.

Definition 3. (Matured Representations) *A matured representation Ψ is defined as a representation that is composed of a set of* **Q***-conjugacy representations. This is symbolically represented as follows.*

$$\Psi = \sum_{i=1}^{s} \alpha_i \hat{\Theta}_i, \tag{84}$$

where each α_i is a rational integer.

Note that the right-hand side of Eq. 84 represents a matrix in which α_i of $\hat{\theta}_i$ are placed as diagonal elements. A matured representation may be an irreducible representation or a reducible representation; the former case can occur if a **Q**-conjugacy character table is identical with the corresponding character table.

Each **Q**-conjugacy representation $\hat{\theta}_i$ affords a **Q**-conjugacy character $\hat{\theta}_i$. Hence, the matured representation of Def. 3 affords a rational character ψ that is a linear combination of the **Q**-conjugacy characters $\hat{\theta}_i$ $(i = 1,2,\cdots,s)$.

$$\psi = \sum_{i=1}^{s} \alpha_i \hat{\theta}_i, \tag{85}$$

where each α_i is a rational integer. The character ψ is called a *matured character*. When a matured character is represented by row vector $\psi = (\psi_j)$, Eqs. 82 and 85 indicate that the coefficients of Eq. 85 are easily obtained by solving linear equations,

$$\psi_j = \sum_{i=1}^{s} \alpha_i \hat{\theta}_{ij} \ (j = 1,2,\cdots,s). \tag{86}$$

These equations are summarized into a theorem by using matrix expressions.

Theorem 5. *When the coefficients α_i for a matured representation (Eq. 85) are collected to form a row vector,*

$$\mathbf{A} = (\alpha_1 \alpha_2 \cdots \alpha_s) \tag{87}$$

the linear equations (Eq. 86) can be written as a matrix expression,

$$\psi = \mathbf{A}D, \tag{88}$$

or equivalently

$$\mathbf{A} = \psi D^{-1}, \tag{89}$$

where the symbol ψ denotes a row vector $\psi = (\psi_j)$ and the symbol D is a **Q***-conjugacy character table (Eq. 83).*

It is worthwhile to mention the relationship among three related concepts, i.e., markaracters, matured characters, and rational characters. Note that markaracters and matured characters are dominant-class (or **Q**-conjugacy-class) functions, while rational characters are class functions.

Let us first examine the relationship between markaracters[21] and rational characters, the latter of which are controled by Artin's theorem.[26] Artin's theorem indicates that any rational character ϕ is represented by

$$\phi = \sum_{i=1}^{s} \frac{a_i}{|\mathbf{G}|} \varphi_i^{\mathbf{G}}, \tag{90}$$

where all a_i are rational integers and $\varphi_i^{\mathbf{G}}$ is the induced character derived from φ_i (the 1-character of a cyclic subgroup $\mathbf{G}_i$). On the other hand, any markaracter is represented by

$$\mathbf{X} = \sum_{i}^{s} \tilde{\alpha}_i \mathbf{G}(/\mathbf{G}_i), \tag{91}$$

as shown in Eq. 7 of Ref. 21. Since $\varphi_i^{\mathbf{G}}$ in Eq. 90 (based on a linear representation) can be regarded as being equivalent to $\mathbf{G}(/\mathbf{G}_i)$ (based on a permutation representation), Eq. 91 (or equivalently Theorem 1 of Ref. 21) is considered to be a special case of Artin's theorem (Eq. 90). However, it should be noted that markaracters are dominant-class functions, while characters are class functions.

Let us next examine the relationship between matured characters and rational characters. Compare Eq. 85 with Eq. 90. The coefficients α_i in Eq. 85 (for a matured character) are rational integers, while the coefficients $a_i/|\mathbf{G}|$ in Eq. 90 (for a rational character) are rational numbers. In the present situation, matured characters are contained in rational characters. However, there naturally emerges a conjecture that matured characters (dominant-class functions) are eqivalent to rational characters (class functions).

3 Combinatorial Enumeration

3.1 Characteristic Monomials. A **Q**-conjugacy representation $\hat{\theta}_i (i = 1,2,\cdots,s)$ is subduced into a cyclic subgroup $\mathbf{G}_j$ to give a linear combination of the **Q**-conjugacy representations $\Gamma_k^{(j)}$ $(k = 1,2,\cdots,v_j)$ for $\mathbf{G}_j$ represented by

$$\hat{\theta}_i \downarrow \mathbf{G}_j = \sum_{k=1}^{v_j} \beta_{jk} \Gamma_k^{(j)} \tag{92}$$

for $i = 1,2,\cdots,s$ and $j = 1,2,\cdots,s$. The coefficients β_{jk} are obtained by solving linear equations concerning the corresponding **Q**-conjugacy characters $\gamma_k^{(j)}$. Thus, we have

$$\hat{\theta}_i \downarrow \mathbf{G}_j = \sum_{k=1}^{v_j} \beta_{jk} \gamma_k^{(j)}, \tag{93}$$

where the symbol $\hat{\theta}_i \downarrow \mathbf{G}_j$ denotes a row vector that contains the elements for $\mathbf{G}_j$ selected from the $\hat{\theta}_i$ row in the **Q**-conjugacy character table of **G**. Once the characteristic monomials for the subgroup $\mathbf{G}_j$ are obtained to be $Z(\Gamma_k^{(j)}; s_d)$, these yield the characteristic monomial for $\hat{\boldsymbol{\Theta}}_i \downarrow \mathbf{G}_j$:

$$Z(\hat{\boldsymbol{\Theta}}_i \downarrow \mathbf{G}_j; s_d) = \prod_{k=1}^{\nu_j} \left(Z(\Gamma_k^{(j)}; s_d) \right)^{\beta_{jk}} \tag{94}$$

for $i = 1, 2, \cdots, s$ and $j = 1, 2, \cdots, s$, where the powers β_{jk} are given in Eq. 93. As a result, we are able to collect these monomials, giving the characteristic monomial table of **G** as an $s \times s$ matrix.

Example 1. In the first step of this example, we shall obtain the characteristic monomial table of cyclic groups. Let us examine a cyclic group $\mathbf{C}_p$ in which p is a prime number. We start from its markaracter table (Table 2, left) and dominant USCI table (Table 2, right) as well as its **Q**-conjugacy character table (Table 3, left). By comparison between Table 2 (left) and Table 3 (left), we easily obtain

$$\mathbf{C}_p(/\mathbf{C}_1) = A + B$$
$$\mathbf{C}_p(/\mathbf{C}_p) = A,$$

which give

$$A = \mathbf{C}_p(/\mathbf{C}_p)$$
$$B = \mathbf{C}_p(/\mathbf{C}_1) - \mathbf{C}_p(/\mathbf{C}_p).$$

Thereby we obtain s_1 for A and $s_1^{-1}s_p$ for B by using the data of the dominant USCI table for $\mathbf{C}_p$ (Table 2, right). These monomials are collected to give the characteristic monomial table for $\mathbf{C}_p$ (Table 3, right).

In the 2nd step, we will obtain the characteristic monomial table of finite groups. Let us consider the group $\mathbf{D}_5$. The subduction $\mathbf{D}_5 \downarrow \mathbf{C}_2$ is accomplished by selecting the corresponding columns from Table 1. The resulting 3×2 matrix is multiplied by the inverse of the **Q**-conjugacy character table of $\mathbf{C}_2$ to give

Table 2. Markaracter Table (left) Dominant USCI Table (right) for $\mathbf{C}_p$ Where p is a Prime Number

$\mathbf{C}_p$	$\mathbf{C}_1$	$\mathbf{C}_p$
$\mathbf{C}_p(/\mathbf{C}_1)$	p	0
$\mathbf{C}_p(/\mathbf{C}_p)$	1	1

$\mathbf{C}_p$	$\mathbf{C}_1$	$\mathbf{C}_p$
$\mathbf{C}_p(/\mathbf{C}_1)$	s_1^p	s_p
$\mathbf{C}_p(/\mathbf{C}_p)$	s_1	s_1
N_j	$\frac{1}{p}$	$\frac{p-1}{p}$

Table 3. **Q**-Conjugacy Character Table (left) and Characteristic Monomial Table (right) for $\mathbf{C}_p$ Where p Is a Prime Number

$\mathbf{C}_p$	$\mathbf{C}_1$	$\mathbf{C}_p$
A	1	1
B	$p-1$	-1

$\mathbf{C}_p$	$\mathbf{C}_1$	$\mathbf{C}_p$
A	s_1	s_1
A	s_1^{p-1}	$s_1^{-1}s_p$
N_j	$\frac{1}{p}$	$\frac{p-1}{p}$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 2 \end{pmatrix} \tag{95}$$

The resulting matrix means that

$$A_1 \downarrow \mathbf{C}_2 = A,$$
$$A_2 \downarrow \mathbf{C}_2 = B,$$
$$E \downarrow \mathbf{C}_2 = 2A + 2B.$$

These results give the following characteristic monomials:

$$Z(A_1 \downarrow \mathbf{C}_2; s_d) = s_1,$$
$$Z(A_1 \downarrow \mathbf{C}_2; s_d) = s_1^{-1}s_2,$$
$$Z(E \downarrow \mathbf{C}_2; s_d) = (s_1)^2(s_1^{-1}s_2)^2 = s_2^2,$$

where we use the characteristic monomials of $\mathbf{C}_2$ collected in the rightmost column of Table 3 (right) where $p = 2$.

Similarly, the subduction $\mathbf{D}_5 \downarrow \mathbf{C}_5$ is accomplished by selecting the corresponding columns from Table 1, which is multiplied by the inverse of the **Q**-conjugacy character table of $\mathbf{C}_5$:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{4}{5} & -\frac{1}{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{96}$$

The matrix in the right-hand side means that

$$A_1 \downarrow \mathbf{C}_5 = A,$$
$$A_2 \downarrow \mathbf{C}_5 = A,$$
$$E \downarrow \mathbf{C}_5 = E$$

The **Q**-conjugacy representations in each equation give the following characteristic monomials:

$$Z(A_1 \downarrow \mathbf{C}_5; s_d) = s_1,$$
$$Z(A_2 \downarrow \mathbf{C}_5; s_d) = s_1,$$
$$Z(E \downarrow \mathbf{C}_5; s_d) = s_1^{-1}s_5,$$

where we use the characteristic monomials of $\mathbf{C}_5$ collected in the rightmost column of Table 3 (right) where $p = 5$.

The characteristic monomials described above are collected to give the characteristic monomial table for $\mathbf{D}_5$ (Table 4).

3.2 Applications of Characteristic Monomials. Such characteristic monomial tables defined in the preceding subsection are effective to solve combinatorial enumeration of isomers. We now consider isomer enumeration based on a skeleton belonging to the **G** group. Suppose that a set of n positions of the skeleton of symmetry **G** is governed by a permutation representation **P**, which is associated with a

Table 4. Characteristic Monomial Table for $\mathbf{D}_5$

	$\mathbf{C}_1$	$\mathbf{C}_2$	$\mathbf{C}_5$
A_1	s_1	s_1	s_1
A_2	s_1	$s_1^{-1}s_2$	s_1
E	s_1^4	s_2^2	$s_1^{-1}s_5$
N_j	$\frac{1}{10}$	$\frac{1}{2}$	$\frac{2}{5}$

matured representation Ψ. These positions are occupied by n of the ligands selected from a ligand set:

$$\mathbf{Y} = \{Y_1, Y_2, \cdots, Y_{|\mathbf{Y}|}\}. \tag{97}$$

The resulting isomer contains ν_ι of ligands Y_ι ($\iota = 1, 2, \cdots, |\mathbf{Y}|$) satisfying a partition:

$$[\nu] : \nu_1 + \nu_2 + \cdots + \nu_{|\mathbf{Y}|} = n, \tag{98}$$

This partition corresponds to the weight (molecular formula) represented by

$$W_\nu = \prod_{\iota=1}^{|\mathbf{Y}|} Y_\iota^{\nu_\iota}. \tag{99}$$

The matured representation Ψ corresponding to $\mathbf{P}$ is reduced into a set of $\mathbf{Q}$-conjugacy representations (Eq. 84). The multiplicities (α_i) are obtained by means of Theorem 5 by placing $\psi = FPV_{\mathbf{P}}$ (the number of fixed points of the skeleton).

By using the multiplicities appearing in Eq. 84 along with characteristic monomials $Z(\hat{\theta}_i \downarrow \mathbf{G}_j; s_d)$ for $\mathbf{G}$ (Eq. 94), we can define a subduced cycle index (SCI):

$$SCI(\mathbf{P} \downarrow \mathbf{G}_j; s_d) = \prod_{i=1}^{s} \left(Z(\hat{\theta}_i \downarrow \mathbf{G}_j; s_d) \right)^{\alpha_i}. \tag{100}$$

By starting from Eq. 100, we have the definition of a cycle index (CI):

$$\begin{aligned} CI(\mathbf{P}; s_d) &= \sum_{i=1}^{s} N_j SCI(\mathbf{P} \downarrow \mathbf{G}_j; s_d) \\ &= \sum_{j=1}^{s} N_j \prod_{i=1}^{s} \left(Z(\hat{\theta}_i \downarrow G_j; s_d) \right)^{\alpha_i}. \end{aligned} \tag{101}$$

where the coefficient N_j is equal to $\varphi(|\mathbf{G}_j|)/|\mathbf{N}_{\mathbf{G}}(\mathbf{G}_j)|$ ($= |\mathbf{K}_j|/|\mathbf{G}|$). The CI (Eq. 101) is applied to combinatorial enumeration, as shown in the following theorem.

Theorem 6. *A generating function for the total number* A_ν *of isomers having the weight* W_ν *(Eq. 99) is represented by*

$$\sum_{[\nu]} A_\nu W_\nu = CI(\mathbf{P}; s_d), \tag{102}$$

where

$$s_d = \sum_{\iota=1}^{|\mathbf{Y}|} Y_\iota^d. \tag{103}$$

This theorem gives enumeration results equivalent to Pólya's theorem or the USCI approach, though the definition of the cycle index (CI) is different from that of Pólya's theorem or from that of the USCI approach.

Example 2. Let us consider planar cyclopentane (**1**) as a skeleton (Chart 1), where some of the 10 hydrogens are replaced by halogen atoms (Y). For simplicity's sake,

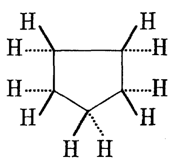

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Chart 1.

we adopt $\mathbf{D}_5$ in place of the full symmetry ($\mathbf{D}_{5h}$) of the cyclopentane. This treatment means that each of enantiomers is counted distinctly. Suppose that the 10 hydrogens are governed by a permutation representation ($\mathbf{P}$) on the action of $\mathbf{D}_5$. Then, the corresponding fixed-point vector ($FPV_{\mathbf{P}}$) is calculated to be (10,0,0). The FPV is multiplied by the inverse of the $\mathbf{Q}$-conjugacy character table of $\mathbf{D}_5$ (Table 1, right) to give

$$(10, 0, 0) \begin{pmatrix} \frac{1}{10} & \frac{1}{10} & \frac{1}{5} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{2}{5} & \frac{2}{5} & -\frac{1}{5} \end{pmatrix} = (1, 1, 2) \tag{104}$$

This means that the $\mathbf{P}$ is reduced into a linear combination of $\mathbf{Q}$-conjugacy representations as follows:

$$\mathbf{P} = A_1 + A_2 + 2E \tag{105}$$

Hence, the SCIs for this case are calculated from the data of the characteristic monomial table of $\mathbf{D}_5$ (Table 4) in the light of Eq. 100:

$$\begin{aligned} SCI(\mathbf{P} \downarrow \mathbf{C}_1; s_d) &= (s_1)(s_1)(s_1^4)^2 = s_1^{10} \\ SCI(\mathbf{P} \downarrow \mathbf{C}_2; s_d) &= (s_1)(s_1^{-1}s_2)(s_2^2)^2 = s_2^5 \\ SCI(\mathbf{P} \downarrow \mathbf{C}_5; s_d) &= (s_1)(s_1)(s_1^{-1}s_5)^2 = s_5^2 \end{aligned}$$

According to Eq. 101, these SCIs are collected to give a CI:

$$f = CI(\mathbf{P}; s_d) = \frac{1}{10}s_1^{10} + \frac{1}{2}s_2^5 + \frac{2}{5}s_5^2, \tag{106}$$

where the coefficients are adopted from the bottom row of Table 4. The figure-inventory,

$$s_d = 1 + y^d, \tag{107}$$

is introduced into the CI to give a generating function:

$$\begin{aligned} f &= \frac{1}{10}(1+y)^{10} + \frac{1}{2}(1+y^2)^5 + \frac{2}{5}(1+y^5)^2 \\ &= 1 + (y+y^9) + 7(y^2+y^8) + 12(y^3+y^7) + 26(y^4+y^6) + 26y^5 \end{aligned} \tag{108}$$

The coefficient of the term y^k represents the number of isomers with k of halogens. For illustrating the result, Fig. 1 depicts 7 disubstituted isomers, which correspond to the term $7y^2$. Note that Fig. 1 contains two enantiomers of *trans*-1,2-disubstituted cyclopentane (in the 2nd row) as well as those of *trans*-1,3-disubstituted cyclopentane (in the bottom row), since $\mathbf{D}_5$ is used in place of $\mathbf{D}_{5h}$.

4 Conclusion

We have proposed the following concepts for the purpose of clarifying the relationship between character tables and mark tables for finite groups:

1. the maturity of an irreducible representation,
2. the inherent automorphism of a finite group,
3. the inner portion of the inherent automorphism,
4. the inherent set of irreducible characters,
5. $\mathbf{Q}$-conjugacy representations and characters,
6. $\mathbf{Q}$-conjugacy character tables, and
7. matured representations and characters.

Thereby, a character table for a finite group is shown to be

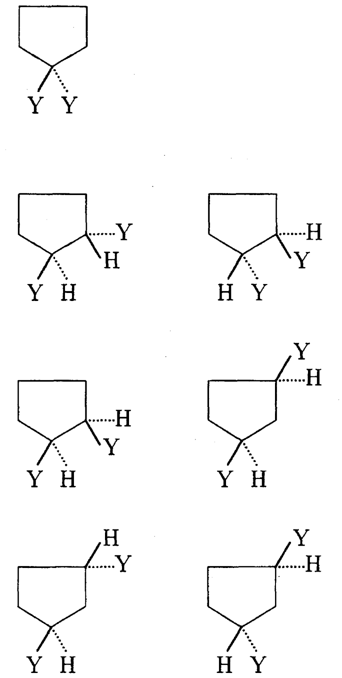

Fig. 1. Seven disubstituted isomers derived from **1**.

transformed into a more concise form (**Q**-conjugacy character table). Moreover, the latter yields a characteristic monomial table for a finite group, which is effective for combinatorial enumeration of isomers.

Appendix A. Automorphism of Cyclic Groups

This appendix is devoted to show that automorphisms of a cyclic group construct an automorphism group. The product of two permutations represented by $P^{(t)}$ and $P^{(s)}$ (Eq. 4) satisfies

$$P^{(t)}P^{(s)} = \begin{pmatrix} \mathbf{H} \\ \mathbf{H}^{(t)} \end{pmatrix} \begin{pmatrix} \mathbf{H} \\ \mathbf{H}^{(s)} \end{pmatrix} = \begin{pmatrix} \mathbf{H}^{(s)} \\ \mathbf{H}^{(ts)} \end{pmatrix} \begin{pmatrix} \mathbf{H} \\ \mathbf{H}^{(s)} \end{pmatrix} = \begin{pmatrix} \mathbf{H} \\ \mathbf{H}^{(ts)} \end{pmatrix} = P^{(ts)}. \tag{109}$$

Hence, the **P** is concluded to be a group, which can be regarded as Aut **H** for the cyclic group **H**. The process shown in Eq. 109 implies that each permutation $P^{(s)}$ corresponds to s, which constructs a group $\mathbf{Z}_n^*$ when s is coprime to n. The group $\mathbf{Z}_n^*$ is a group of irreducible residue classes (mod n). Note that $|\mathbf{Z}_n^*| = \varphi(n)$, where $n = |\mathbf{H}|$. Hence, we arrive at a theorem.

Theorem 7. *Let* **P** *be the automorphic group for the cyclic group* **H** *of order n. Then we have*

$$\mathbf{P} \cong \mathbf{Z}_n^* \tag{110}$$

Let us consider the inverse elements of **H** (Eq. 1). The resulting ordered set is designated to be

$$\mathbf{H}^{(-1)} = \left\{\tilde{h}^{-1}, \tilde{h}^{-2}, \cdots, \tilde{h}^{-r}, \cdots, \tilde{h}^{-n}\right\}. \tag{111}$$

The correspondence $\tilde{h} \rightarrow \tilde{h}^{-1}$ provides a permutation:

$$P^{(-1)} = \begin{pmatrix} \mathbf{H} \\ \mathbf{H}^{(-1)} \end{pmatrix} = \begin{pmatrix} \tilde{h}^1 & \tilde{h}^2 & \cdots & \tilde{h}^n \\ \tilde{h}^{-1} & \tilde{h}^{-2} & \cdots & \tilde{h}^{-n} \end{pmatrix}. \tag{112}$$

Since we have $(P^{(-1)})^2 = I$, the set,

$$\tilde{\mathbf{P}} = \{P^{(1)}(=I), P^{(-1)}\}, \tag{113}$$

is a group of order 2, which is a subgroup of Aut **H**. This is summarized as a corollary:

Corollary 1. *The automorphism group* (Aut **H**) *of a cyclic group* **H** *contains a subgroup of order 2.*

Aut **H** of a cyclic group **H** provides a partition of the elements of **H**. The resulting orbits relate to the primitive roots of 1.

The behavior of the elements of a cyclic group is governed by the following theorem.

Theorem 8. *Let us remember that the automorphism group* Aut **H** *is regarded as a permutation group* **P** *acting on a cyclic group* **H**, *which is regarded as an ordered set* (*Eq. 1*). *Thereby, the set* **H** *is partitioned into orbits, which correspond to the respective dominant classes of* **H**.

For the dominant classes of a cyclic group, see the preceding paper.[23] Each dominant class of **H** corresponds to a distinct cyclic subgroup, which is in turn characterized by an n_i-th root of 1 through the generator of the subgroup, where n_i represents a divisor of $n(=|\mathbf{H}|)$. In particular, the dominant class that consists of the generators of the group **H** corresponds to the n-th roots of 1. This fact is another expression of Theorem 7. Moreover, the dominant class is characterized by the coset representation $\mathbf{P}(/\{\mathrm{I}\})$, the degree of which is equal to $|\mathbf{P}|$ $(=\varphi(|\mathbf{H}|))$. It should be added that the latter dominant class is identical with $\mathbf{K}_i \cap \mathbf{H}$ if **H** is a cyclic subgroup of the group **G**.

In virtue of Theorem 8, we can independently discuss the dominant class that consists of the generators of the group **H**. In the following discussions, we pay special attention to such a dominant class, even when we deal with **H**.

Appendix B

We show the sets of **Q**-conjugacy character table and a markaracter table for several point groups (Tables 5, 6, 7, and 8).

Table 5. **Q**-Conjugacy Character Table (left) and Markaracter Table (right) for $\mathbf{C}_{4h}$

	$\mathbf{C}_1$	$\mathbf{C}_2$	$\mathbf{C}_s$	$\mathbf{C}_i$	$\mathbf{C}_4$	$\mathbf{C}_4$
A_g	1	1	1	1	1	1
B_g	1	1	1	1	−1	−1
E_g	2	−2	−2	2	0	0
A_u	1	1	−1	−1	1	−1
B_u	1	1	−1	−1	−1	1
E_u	2	−2	2	−2	0	0

	$\mathbf{C}_1$	$\mathbf{C}_2$	$\mathbf{C}_s$	$\mathbf{C}_i$	$\mathbf{C}_4$	$\mathbf{C}_4$
$\mathbf{C}_{4h}(/\mathbf{C}_1)$	8	0	0	0	0	0
$\mathbf{C}_{4h}(/\mathbf{C}_2)$	4	4	0	0	0	0
$\mathbf{C}_{4h}(/\mathbf{C}_s)$	4	0	4	0	0	0
$\mathbf{C}_{4h}(/\mathbf{C}_i)$	4	0	0	4	0	0
$\mathbf{C}_{4h}(/\mathbf{C}_4)$	2	2	0	0	2	0
$\mathbf{C}_{4h}(/\mathbf{C}_4)$	2	2	0	0	0	2

Table 6. **Q**-Conjugacy Character Table (left) and Markaracter Table (right) for $\mathbf{C}_{6h}$

	$\mathbf{C}_1$	$\mathbf{C}_2$	$\mathbf{C}_s$	$\mathbf{C}_i$	$\mathbf{C}_3$	$\mathbf{C}_6$	$\mathbf{C}_{3h}$	$\mathbf{C}_{3i}$
A_g	1	1	1	1	1	1	1	1
B_g	1	−1	−1	1	1	−1	−1	1
E_{1g}	2	−2	−2	2	−1	1	1	−1
E_{2g}	2	2	2	2	−1	−1	−1	−1
A_u	1	1	−1	−1	1	1	−1	−1
B_u	1	−1	1	−1	1	−1	1	−1
E_{1u}	2	−2	2	−2	−1	1	−1	1
E_{2u}	2	2	−2	−2	−1	−1	1	1

	$\mathbf{C}_1$	$\mathbf{C}_2$	$\mathbf{C}_s$	$\mathbf{C}_i$	$\mathbf{C}_3$	$\mathbf{C}_6$	$\mathbf{C}_{3h}$	$\mathbf{C}_{3i}$
$\mathbf{C}_{6h}(/\mathbf{C}_1)$	12	0	0	0	0	0	0	0
$\mathbf{C}_{6h}(/\mathbf{C}_2)$	6	6	0	0	0	0	0	0
$\mathbf{C}_{6h}(/\mathbf{C}_s)$	6	0	6	0	0	0	0	0
$\mathbf{C}_{6h}(/\mathbf{C}_i)$	6	0	0	6	0	0	0	0
$\mathbf{C}_{6h}(/\mathbf{C}_3)$	4	0	0	0	4	0	0	0
$\mathbf{C}_{6h}(/\mathbf{C}_6)$	2	2	0	0	2	2	0	0
$\mathbf{C}_{6h}(/\mathbf{C}_{3h})$	2	0	2	0	2	0	2	0
$\mathbf{C}_{6h}(/\mathbf{C}_{3i})$	2	0	0	2	2	0	0	2

Table 7. **Q**-Conjugacy Character Table (left) and Markaracter Table (right) for $\mathbf{D}_{4d}$

	$\mathbf{C}_1$	$\mathbf{C}_2$	$\mathbf{C}'_2$	$\mathbf{C}_s$	$\mathbf{C}_4$	$\mathbf{S}_8$
A_1	1	1	1	1	1	1
B_2	1	1	−1	−1	1	1
B_1	1	1	1	−1	1	−1
B_2	1	1	−1	1	1	−1
E_2	2	2	0	0	−2	0
E_1+E_3	4	−4	0	0	0	0

	$\mathbf{C}_1$	$\mathbf{C}_2$	$\mathbf{C}'_2$	$\mathbf{C}_s$	$\mathbf{C}_4$	$\mathbf{S}_8$
$\mathbf{C}_{4d}(/\mathbf{C}_1)$	16	0	0	0	0	0
$\mathbf{C}_{4d}(/\mathbf{C}_2)$	8	8	0	0	0	0
$\mathbf{C}_{4d}(/\mathbf{C}'_2)$	8	0	2	0	0	0
$\mathbf{C}_{4d}(/\mathbf{C}_s)$	8	0	0	2	0	0
$\mathbf{C}_{4d}(/\mathbf{C}_4)$	4	4	0	0	4	0
$\mathbf{C}_{4d}(/\mathbf{S}_s)$	2	2	0	0	2	2

Table 8. **Q**-Conjugacy Character Table (left) and Markaracter Table (right) for $\mathbf{T}_h$

	$\mathbf{C}_1$	$\mathbf{C}_2$	$\mathbf{C}_s$	$\mathbf{C}_i$	$\mathbf{C}_3$	$\mathbf{C}_6$
A_g	1	1	1	1	1	1
A_u	1	1	−1	−1	1	−1
E_g	2	2	2	2	−1	−1
E_u	2	2	−2	−2	−1	1
T_g	3	−1	−1	3	0	0
T_u	3	−1	1	−3	0	0

	$\mathbf{C}_1$	$\mathbf{C}_2$	$\mathbf{C}_s$	$\mathbf{C}_i$	$\mathbf{C}_3$	$\mathbf{C}_6$
$\mathbf{T}_h(/\mathbf{C}_1)$	24	0	0	0	0	0
$\mathbf{T}_h(/\mathbf{C}_2)$	12	4	0	0	0	0
$\mathbf{T}_h(/\mathbf{C}_s)$	12	0	4	0	0	0
$\mathbf{T}_h(/\mathbf{C}_i)$	12	0	0	12	0	0
$\mathbf{T}_h(/\mathbf{C}_3)$	8	0	0	0	2	0
$\mathbf{T}_h(/\mathbf{C}_6)$	4	0	0	4	1	1

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